Bruhat order and full symmetric Toda flow

Georgy Sharygin
joint work with Yu. Chernyakov (ITEP)
A. Sorin (JINR)

MSU (Moscow State University) and
ITEP (Institute for Theoretical and Experimental Physics)

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Classical non-periodic Toda system (Toda chain) consists of n particles on the line with exponential interactions between neighbours. The Hamiltonian of this system is given by

$$H = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} \exp(q_i - q_{i+1}), \tag{1}$$

where p_i is the impulse of the i^{th} particle and q_i is its coordinate. The Poisson structure on the phase space (p_i, q_i) has the standard form

$$\{p_i, q_j\} = \delta_{ij},\ \{p_i, p_j\} = 0,\ \{q_i, q_j\} = 0. \tag{2}$$

The evolution of the system is given by the usual Hamiltonian equations:

$$\dot{p}_i = \{H, q_i\},\ \dot{q}_i = -\{H, p_i\}.$$

If we make the following ansatz

$$b_i = p_i, \quad a_i = \exp\frac{1}{2}(q_i - q_{i+1}), \tag{3}$$

the Hamiltonian will take the form

$$H = \sum_{i=1}^{n}\frac{1}{2}b_i^2 + \sum_{i=1}^{n-1}a_i^2. \tag{4}$$

Observe, that there are only $n-1$ variables a_i. The Poisson structure (2) turns into

$$\{b_i, a_{i-1}\} = -a_{i-1},\ \{b_i, a_i\} = a_i. \tag{5}$$

All the other brackets of coordinates a_i and b_j are equal to zero.

In these coordinates it is easy to find the Lax representation: the Hamilton equations are equivalent to the following equation

$$\dot{L} = [L, A] \tag{6}$$

Here

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & b_n \end{pmatrix}, \; A = \begin{pmatrix} 0 & -a_1 & 0 & \dots & 0 \\ a_1 & 0 & -a_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 \end{pmatrix} \tag{7}$$

So this system is integrable: the commuting integrals can be taken in the form $Tr(L^k),\ k = 1, \dots, n$. In fact, $Tr(L)$ is a Kasimir function, so one can set its value equal to 0. The eigenvalues λ_i of L do not change with time, but their order does. It was shown by Moser, that when $\lambda_i \neq \lambda_j,\ t \to -\infty$, the matrix L converges to diagonal matrix in which the order of λ_i is increasing, and when $t \to +\infty$, we obtain diagonal matrix with decreasing λ_i.

Full symmetric Toda system is a straightforward generalization of the three-diagonal Toda system. It is the system of differential equations on the space of traceless symmetric matrices $Symm_n$, given by the following Lax equation

$$\dot{L} = [L, A], \tag{8}$$

where L is the varying matrix and $A = A(L)$ its anti-symmetrisation, i.e.

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}, \ A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix}.$$

As a matter of fact, this system is Hamiltonian and integrable. The Hamiltomnian structure is induced from the isomorphisms:

$$\mathfrak{sl}_n = \mathfrak{so}_n \oplus \mathfrak{b}_n^+,\ \mathfrak{sl}_n^* = (\mathfrak{b}_n^+)^* \oplus (\mathfrak{so}_n)^*,$$
$$(\mathfrak{b}_n^+)^* \cong (\mathfrak{so}_n)^\perp = Symm_n,\ (\mathfrak{so}_n)^* \cong (\mathfrak{b}_n^+)^\perp = \mathfrak{n}_n^+,$$

given by the Killing form. Thus, we can use the isomorphism $Symm_n \cong (\mathfrak{b}_n^+)^*$ (where $\mathfrak{b}_n^+$ denotes the Borel algebra of upper-triangular matrices) to pull the Poisson structure to $Symm_n$. The Hamiltonian function is given by $H(L) = Tr(L^2)$.

One can show that this system is again integrable. However, this time the standard set of first integrals $H_i(L) = Tr(L^i),\ i = 1, \dots, n$ does not suffice, since dimension of the space is $\frac{n(n+1)}{2} - 1$, much more than n, so one has to look for additional integrals of the motion. This will be the subject of the talk by Yu. Chernyakov. I here will need the following few properties, pertaining to the integrability.

The question, that we address in this talk is:

Describe the asymptotic behaviour of the full symmetric Toda system, i.e. let $t \to \pm\infty$, is it true, that the matrix L converges to a diagonal matrix? If yes, what is the order of eigenvalues of these matrices? In what case two matrices with different ordering of the eigenvalues are connected by a trajectory? How many different trajectories there exist?

This questions have been addressed in the paper

P.Fre, A.Sorin, The arrow of time and the Weyl group: all supergravity billiards are integrable, arXiv:0710.1059

in the case $n = 3$. They used explicit formulas for the solutions of the Toda system to this end.

The main theorem is

Theorem

Let all the eigenvalues λ_i of L be different. Then the matrix L converges to a diagonal matrix with diagonal entries, equal to λ_i when $t \to \pm\infty$. The order of the eigenvalues is arbitrary. If we fix the indices of λ_i so that $\lambda_i < \lambda_{i+1}$ for all $i = 1, \dots, n-1$, then all the asymptotic matrices can be identified with the elements of the permutation group S_n. Then any two diagonal matrices are connected by a trajectory if and only if the corresponding permutations are comparable in Bruhat order on S_n. Moreover, the dimension of the manifold, spanned by the trajectories, connecting w and w' is equal to $|l(w) - l(w')| + 1$.

Below we shall explain the definition of Bruhat order and other technical matters.

We begin with the following observation: for every symmetric matrix L there exist a special orthogonal matrix Ψ, such, that

$$L = \Psi\Lambda\Psi^{-1}, \text{ where } \Lambda = diag(\lambda_1, \ldots, \lambda_n).$$

The matrix Ψ is not unique, but if $\lambda_i \neq \lambda_j,\ i \neq j$, it is determined up to a finite set of diagonal matrices with entries, equal to ± 1. Thus, we can assume, that this matrix Ψ is locally unique. Then the equation (8) can be rewritten in the terms of Ψ:

$$\frac{d\Psi}{dt} = -A\Psi, \tag{9}$$

where we put $A = A(\Psi)$ to be the composition of $A(L)$ with the expression $L = \Psi\Lambda\Psi^{-1}$:

$$A(\Psi) = (\Psi\Lambda\Psi^{-1})_+ - (\Psi\Lambda\Psi^{-1})_-. \tag{10}$$

Thus assuming that $\lambda_i \neq \lambda_j$ (which we shall always do below) we can speak about the evolution of L in the terms of the evolution of the matrix Ψ in $SO_n(\mathbb{R})$, given by the equation (9).

More accurately, let T_n be the group of diagonal matrices with ± 1 on diagonal and determinant equal to 1. Then Ψ can be regarded as the point in $SO_n(\mathbb{R})/T_n$, which is isomorphic to the flag manifold $Fl_n(\mathbb{R})$. Recall, that a (full) flag $E.$ in $\mathbb{R}^n$ is a collection of hyperplanes

$$\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{R}^n$$

in $\mathbb{R}^n$, such that $\dim E_i = i$.

Thus, the Toda system can be regarded as a system on the flag space. In what follows we shall stick to this point of view. From now on we shall fix the set of eigenvalues $\lambda_i \neq \lambda_j,\ i \neq j$.

It is convenient to describe the integrals of the Toda system in the terms of the system on $Fl_n(\mathbb{R})$ that we have just constructed. In effect, this is the approach, used by Yu. Chernyakov and A. Sorin. In fact, the additional integrals can be chosen to be equal to the rational functions of minors $M_{\frac{i_1\dots i_k}{1\dots k}}$ of matrix Ψ (here we use this notation for the determinants of the submatrices of Ψ, spanned by the intersections of first (or last) k rows and by arbitrary k columns of Ψ). The equations of motion of these minors have the following form

$$\dot{M} = f(\Lambda, \psi)M, \tag{11}$$

where f is some regular function of Λ and ψ.

The level set $M = 0$ of M is an invariant subvariety of the Toda system. We shall call these surfaces the minor surfaces of the Toda flow. For instance in the case $n = 3$ there are six such surfaces, which correspond to six 1×1 submatrices and when $n = 4$ there are eight distinct 1×1 minors and six 2×2 minors, which give invariant surfaces.

Recall, that Morse function on a compact smooth manifold M is a smooth function, with a discrete set of nondegenerate singular ponts, i.e. points x_0, in which $df(x_0) = 0$ and the Hessian matrix $d^2 f(x_0)$ is nondegenerate. One can regard the Hessian $d^2 f(x_0)$ as a quadratic form on $T_{x_0}M$; then the index of this quadratic form (the number of -1s in its canonic form) is called the index of the singular point x_0 of f. It is also usually assumed, that the values of f in its singular points are different.

Let f be a Morse function; for any Riemannian structure g on M, we can define the gradient $\mathrm{grad}\, f$ as the vector field equal to df with raised indices. One readily sees, that this vector field has the same set of singular points as f. One can prove the Morse Lemma: for every singular point x_0 of index k there exists an orthogonal in x_0 coordinate system $\xi^1, \dots, \xi^n$ in a neighbourhood of x_0, s.t.

$$f = f(x_0) - (\xi^1)^2 - \cdots - (\xi^k)^2 + (\xi^{k+1})^2 + \cdots + (\xi^n)^2.$$

One can use these coordinates to describe the behaviour of the trajectories of the gradient vector field. First, there can be no closed trajectories (since the value of f grows along any trajectory); so any trajectory should connect two singular points of the function. Second, in the neighbourhood of a singular point x_0, we have a k-dimensional submanifold, spanned by the trajectories, incoming to x_0; this manifold is called unstable submanifold of the point x_0 denoted $W^u_{x_0}(f)$, and a dimension $n-k$ submanifold, spanned by the trajectories, exiting from x_0, the stable submanifold of the point x_0 denoted $W^s_{x_0}(f)$.

Clearly, two singular points x_0 and x_1 are connected by a trajectory, iff, say $f(x_1) > f(x_0)$ and $W^u_{x_1}(f) \cup W^s_{x_0}(f) \neq \emptyset$, and there are as many trajectories as many points are in this intersection. In general it is not easy to describe this intersection. An important particular case is the case of Morse-Smale system: the pair (f, g) is said to be a Morse-Smale, if all the stable and unstable submanifolds intersect transversally. In this case the dimension is given by the usual dimension counting

$$\dim U \cup V = \dim U + \dim V - \dim M.$$

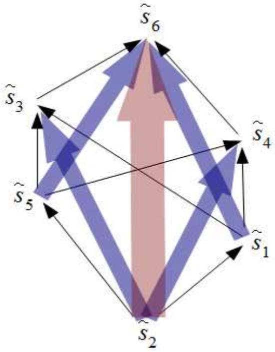
The important property of the Toda system on $SO_n(\mathbb{R})$ is that it is a gradient system of a Morse function, i.e. there is a Morse function $F_n(\Psi)$, $\Psi \in SO_n(\mathbb{R})$, such that its gradient, with respect to certain Riemannian structure on $SO_n(\mathbb{R})$, is equal to $-A(\Psi)\Psi$. In fact the matrix A can be written in the form

$$A = J([L, N]),$$

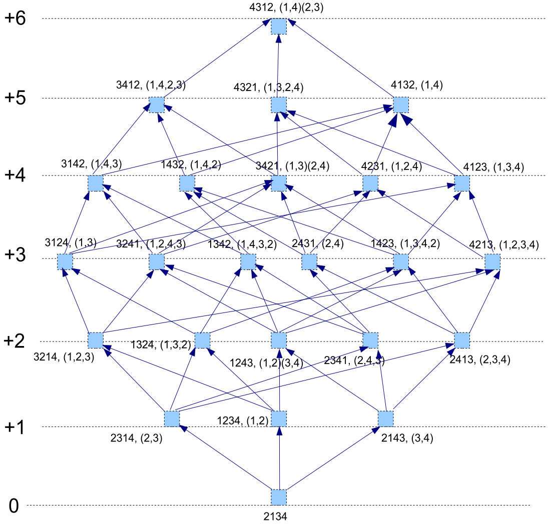
where N is a diagonal matrix, and J – a symmetric linear operator on the space of antisymmetric matrices $\mathfrak{so}_n$, equal to the division by k on the k-th upper- and lower-diagonals of a matrix. The operator J is also used in the definition of the Riemannian structure: one puts

$$\langle A,\, B\rangle_J = \langle A,\, J(B)\rangle = -Tr(AJ(B)),$$

where $A,\, B \in \mathfrak{so}_n$ are arbitrary antisymmetric matrices.



This is is enough to describe the asymptotic behaviour in the low dimensions. First, we notice that $A(\Psi) = 0$ iff the matrix Ψ is (up to the diagonal group T_n) a matrix of permutation. Thus, the first statement of the theorem is proved. For $n = 3$, we have 6 two-dimensional minor surfaces, which intersect over one-dimensional curves. The trajectories flow in the direction in which the values of F_3 grow. Calculating the signs of the Hessians on the surfaces, we obtain the diagramm, drawn here: thin arrows correspond to single trajectories, blue arrows to one-parametric families and the red arrow to a two-parametric family of trajectories. As one can see, this is precisely the Hasse diagramm of the Bruhat order on S_3 (see below). Similar calculations can be performed for $SO_4(\mathbb{R})$ with the same result (see next slide). In order toanswer the question in a general case, we need more delicate constructions.



The asymptotic flows on $SO_4(\mathbb{R})$.

Let w be a permutation on the set $\{1,\dots,n\}$. One defines the height (or length) of w, denoted $l(w)$, as the number of inversions in w:

$$l(w) = \#\{(i,j) \mid 1 \leq i < j \leq n,\ w(i) > w(j)\}. \tag{12}$$

One can now introduce a partial order on S_n, the (weak) Bruhat order as the minimal partial order, generated by the following relations:

$$x < y, \text{ if and only if } y = (i,j)x, \text{ and } l(y) = 1 + l(x). \tag{13}$$

Given a partial order on a finite set, its Hasse diagramm is the graph, whose vertices are theelements of the set, and the edges connect two neighbours w.r.t. the given partial order.

For each permutation w, one defines the corresponding Schubert cell in $Fl_n(\mathbb{R})$ as:

Definition

Schubert cell $X_w \subset Fl_n(\mathbb{R})$ is the set of the flags verifying the following conditions:

$$X_w = \{E_\cdot \in Fl_n \mid \dim(E_p \cap F_q) = \#\{i \mid 1 \le i \le p,\ w(i) \le q\}\ \forall\ 1 \le p, q \le n\}, \tag{14}$$

where flag $E_\cdot = E_1 \subset E_2 \subset \ldots \subset E_n = \mathbb{R}^n$ and $\dim(E_i) = i$, $F_q = \langle e_1, \ldots, e_q \rangle$ is the space, spanned by the first q elements of the basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$.

Schubert variety $\overline{X}_w$ is defined as the closure of the corresponding Schubert cell. One can show that $w < w'$ in Bruhat order, iff $\overline{X}_w \subseteq \overline{X}_{w'}$ (a classical result, to be found in many books). besides this for any $\overline{X}_w$, one can find a set of equations on the elements of $SL_n(\mathbb{R})$ (or $SO_n(\mathbb{R})$), that determine $\overline{X}_w$ (or, rather its preimage in the corresponding group).

In addition to the Schubert cells X_w, one can define dual Schubert cells Ω_w, as the set of those flags $E.$, for which for all p, q we have

$$\dim(E_p \bigcap \tilde{F}_q) = \#\{i \leq p \mid w(i) \geq n + 1 - q\},$$

where $\tilde{F}_q$ is the subspace, spanned by the last q vectors of the basis. Dual Schubert variety is defined similarly, and can also be described by equations. The following three facts about the Schubert cells are taken from literature:

1. $\dim X_w = l(w), \ \dim \Omega_w = \frac{n(n+1)}{2} + 1 - l(w);$
2. $X_w \cup \Omega_u \neq \varnothing$ iff $u < w$ in Bruhat order, then they intersect transversally;
3. $T_w X_w$ is spanned by positive roots of $SL_n(\mathbb{R})$, mapped into negative by w; $T_w \Omega_w$ is spanned by the positive roots of SL_n, mapped into positive by w.

The important steps towards the proof are the following lemmas:

Lemma

Toda flow preserves Schubert cells and dual Schuber cells (in other words, the corresponding vector field is tangent to Schubert cells).

Lemma

If the order of the eigenvalues in Λ *is given by* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, *then the negative (resp. positive) eigenspace of the Hessian of the Morse function* F_n *of at* w *is spanned by those positive roots of* SL_n, *which are mapped into negative (resp. positive) roots by* w.

Both lemmas are proved by straightforward calculations: in case of lemma 2 we use the systems of equations, that determine X_w or Ω_w and show, that they are preserved by the flow. In the case of lemma 3 we calculate the Hessian in the local coordinates, given by the right translation of $\mathfrak{sl}_n$ by w.

Now the proof follows from the previous observations: we see, that restriction of the gradient flow to a (dual) Schubert cell coincides with the gradient flow of the restriction of the function on it (lemma 2), hence the cell X_w (resp. dual cell Ω_w) coincides with the unstable (resp., stable) submanifold of F_n in this point (lemma 3 and the third property of Schubert cells). Thus (see property 2 of the Schubert cells) we must conclude, that the system is Morse-Smale and two points are connected by a trajectory, iff the corresponding permutations are Bruhat-comparable. Finally, the dimansion of the space of trajectories, connecting them is given by the formula, mentioned in thoerem.